Derivatives

8.1 Definition of the derivative: The **derivative** of a function of a real variable measures the sensitivity to change of a quantity (a function or dependent variable) which is determined by another quantity (the independent variable). It is a fundamental tool of calculus. For example, the derivative of the position of a moving object with respect to time is the object's velocity: this measures how quickly the position of the object changes when time is advanced. The derivative measures the *instantaneous* rate of change of the function, as distinct from its *average* rate of change, and is defined as the limit of the average rate of change in the function as the length of the interval on which the average is computed tends to zero.

The derivative of a function at a chosen input value describes the best linear approximation of the function near that input value. In fact, the derivative at a point of a function of a single variable is the slope of the tangent line to the graph of the function at that point.

The notion of derivative may be generalized to functions of several real variables. The generalized derivative is a linear map called the differential. Itsmatrix representation is the Jacobian matrix, which reduces to the gradient vector in the case of real-valued function of several variables.

The process of finding a derivative is called **differentiation**. The reverse process is called *antidifferentiation*. The fundamental theorem of calculusstates that antidifferentiation is the same as integration. Differentiation and integration constitute the two fundamental operations in single-variable calculus.[1]

Differentiation and the derivative

Differentiation is the action of computing a derivative. The derivative of a function $f(x)$ of a variable x is a measure of the rate at which the value of the function changes with respect to the change of the variable. It is called the *derivative* of f with respect to x. If x and y are real numbers, and if the graph of f is plotted against x, the derivative is the slope of this graph at each point.

The simplest case, apart from the trivial case of a constant function, is when y is a linear function of x, meaning that the graph of y divided by x is a line. In this case, $y = f(x) = m x + b$, for real numbers m and b, and the slope m is given by

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x},$$

where the symbol Δ (Delta) is an abbreviation for "change in." This formula is true because

$$y + \Delta y = f(x + \Delta x) = m\,(x + \Delta x) + b = m\,x + m\,\Delta x + b = y + m\,\Delta x.$$

It follows that $\Delta y = m\,\Delta x$.

This gives an exact value for the slope of a line. If the function f is not linear (i.e. its graph is not a line), however, then the change in y divided by the change in x varies: differentiation is a method to find an exact value for this rate of change at any given value of x.

Rate of change as a limit value

Figure 1. The tangent line at (x, $f(x)$)

Figure 2. The secant to curve $y= f(x)$ determined by points (x, $f(x)$) and ($x+h$,$f(x+h)$)

Figure 3. The tangent line as limit of secants

Figure 4. Animated illustration: the tangent line (derivative) as the limit of secants

The idea, illustrated by Figures 1 to 3, is to compute the rate of change as the limit value of the ratio of the differences Δy / Δx as Δx becomes infinitely small.

Notation

Two distinct notations are commonly used for the derivative, one deriving from Leibniz and the other from Joseph Louis Lagrange.

In Leibniz's notation, an infinitesimal change in x is denoted by dx, and the derivative of y with respect to x is written

$$\frac{dy}{dx}$$

suggesting the ratio of two infinitesimal quantities. (The above expression is read as "the derivative of y with respect to x", "d y by d x", or "d y over d x". The oral form "d y d x" is often used conversationally, although it may lead to confusion.)

In Lagrange's notation, the derivative with respect to *x* of a function *f*(*x*) is denoted *f*′(*x*) (read as "f prime of x") or $f_x'(x)$ (read as "f prime x of x"), in case of ambiguity of the variable implied by the derivation. Lagrange's notation is sometimes incorrectly attributed to Newton.

Rigorous definition

The most common approach to turn this intuitive idea into a precise definition is to define the derivative as a limit of difference quotients of real numbers.[2] This is the approach described below.

Let *f* be a real valued function defined in an open neighborhood of a real number *a*. In classical geometry, the tangent line to the graph of the function *f* at *a* was the unique line through the point (*a*, *f*(*a*)) that did *not* meet the graph of *f* transversally, meaning that the line did not pass straight through the graph. The derivative of *y* with respect to *x* at *a* is, geometrically, the slope of the tangent line to the graph of *f* at (*a*, *f*(*a*)). The slope of the tangent line is very close to the slope of the line through (*a*, *f*(*a*)) and a nearby point on the graph, for example (*a* + *h*, *f*(*a* + *h*)). These lines are called secant lines. A value of *h* close to zero gives a good approximation to the slope of the tangent line, and smaller values (in absolute value) of *h* will, in general, give better approximations. The slope *m* of the secant line is the difference between the *y* values of these points divided by the difference between the *x* values, that is,

$$m = \frac{\Delta f(a)}{\Delta a} = \frac{f(a+h) - f(a)}{(a+h) - (a)} = \frac{f(a+h) - f(a)}{h}.$$

Continuity and differentiability

This function does not have a derivative at the marked point, as the function is not continuous there. If y = f(x) is differentiable at a, then f must also be continuous at a. As an example, choose a point a and let f be the step function that returns a value, say 1, for all x less than a, and returns a different value, say 10, for all x greater than or equal to a. f cannot have a derivative at a. If h is negative, then a + h is on the low part of the step, so the secant line from a to a + h is very steep, and as h tends to zero the slope tends to infinity. If h is positive, then a + h is on the high part of the step, so the secant line from a to a + h has slope zero. Consequently the secant lines do not approach any single slope, so the limit of the difference quotient does not exist.

The absolute value function is continuous, but fails to be differentiable at $x = 0$ since the tangent slopes do not approach the same value from the left as they do from the right.
However, even if a function is continuous at a point, it may not be differentiable there. For example, the absolute value function x is continuous at $x = 0$, but it is not differentiable there. If h is positive, then the slope of the secant line from 0 to h is one, whereas if h is negative, then the slope of the secant line from 0 to h is negative one. This can be seen graphically as a "kink" or a "cusp" in the graph at $x = 0$. Even a function with a smooth graph is not differentiable at a point where its tangent is vertical: For instance, the function $y = x1/3$ is not differentiable at $x = 0$.

In summary: for a function f to have a derivative it is necessary for the function f to be continuous, but continuity alone is not sufficient.

Most functions that occur in practice have derivatives at all points or at almost every point. Early in the history of calculus, many mathematicians assumed that a continuous function was differentiable at most points. Under mild conditions, for example if the function is a monotone function or a Lipschitz function, this is true. However, in 1872 Weierstrass found the first example of a function that is continuous everywhere but differentiable nowhere. This example is now known as the Weierstrass function. In 1931, Stefan Banach proved that the set of functions that have a derivative at some point is a meager set in the space of all continuous functions.[4] Informally, this means that hardly any continuous functions have a derivative at even one point.

The derivative as a function

Let f be a function that has a derivative at every point a in the domain of f. Because every point a has a derivative, there is a function that sends the point a to the derivative of f at a. This function is written $f'(x)$ and is called the derivative function or the derivative of f. The derivative of f collects all the derivatives of f at all the points in the domain of f.

Sometimes f has a derivative at most, but not all, points of its domain. The function whose value at a equals $f'(a)$ whenever $f'(a)$ is defined and elsewhere is undefined is also called the derivative of f. It is still a function, but its domain is strictly smaller than the domain of f.

Using this idea, differentiation becomes a function of functions: The derivative is an operator whose domain is the set of all functions that have derivatives at every point of their domain and whose range is a set of functions. If we denote this operator by D, then D(f) is the function $f'(x)$. Since D(f) is a function, it can be evaluated at a point a. By the definition of the derivative function, $D(f)(a) = f'(a)$.

Higher derivatives

Let f be a differentiable function, and let $f'(x)$ be its derivative. The derivative of $f'(x)$ (if it has one) is written $f''(x)$ and is called the second derivative of f. Similarly, the derivative of a second derivative, if it exists, is written $f'''(x)$ and is called the third derivative of f. Continuing this process, one can define, if it exists, the nth derivative as the derivative of the (n-1)th derivative. These repeated derivatives are called higher-order derivatives. The nth derivative is also called the derivative of order n.

If x(t) represents the position of an object at time t, then the higher-order derivatives of x have physical interpretations. The second derivative of x is the derivative of $x'(t)$, the velocity, and by definition this is the object's acceleration. The third derivative of x is defined to be the jerk, and the fourth derivative is defined to be the jounce.

In finance, a derivative is a special type of contract that derives its value from the performance of an underlying entity. This underlying entity can be an asset, index, or interest rate, and is often called the "underlying".[1][2] Derivatives can be used for a number of purposes - including insuring against price movements (hedging), increasing exposure to price movements for speculation or getting access to otherwise hard to trade assets or markets.

Some of the more common derivatives include futures, forwards, swaps, options, and variations of these such as caps, floors, collars, and credit default swaps. Most derivatives are traded over-the-counter (off-exchange) or on an exchange such as the Chicago Mercantile Exchange, while most insurance contracts have developed into a separate industry. Derivatives are one of the three main categories of financial instruments, the other two being equities (i.e. stocks or shares) and debt (i.e. bonds and mortgages).

Basics[edit]

Derivatives are a contract between two parties that specify conditions (especially the dates, resulting values and definitions of the underlying variables, the parties' contractual obligations, and the notional amount) under which payments are to be made between the parties.[3][4] The most common underlying assets include commodities, stocks, bonds, interest rates and currencies, but they can also be other derivatives, which adds another layer of complexity to proper valuation. The components of a firm's capital structure, e.g. bonds and stock, can also be considered derivatives, more precisely options, with the underlying being the firm's assets, but this is unusual outside of technical contexts.

There are two groups of derivative contracts: the privately traded over-the-counter (OTC) derivatives such as swaps that do not go through an exchange or other intermediary, and exchange-traded derivatives (ETD) that are traded through specialized derivatives exchanges or other exchanges.

Derivatives are more common in the modern era, but their origins trace back several centuries. One of the oldest derivatives is rice futures, which have been traded on the Dojima Rice Exchange since the eighteenth century.[6] Derivatives are broadly categorized by the relationship between the underlying asset and the derivative (such as forward, option, swap); the type of underlying asset (such as equity derivatives, foreign exchange derivatives, interest rate derivatives, commodity derivatives, or credit derivatives); the market in which they trade (such as exchange-traded or over-the-counter); and their pay-off profile.

Derivatives may broadly be categorized as "lock" or "option" products. Lock products (such as swaps, futures, or forwards) obligate the contractual parties to the terms over the life of the contract. Option products (such as interest rate caps) provide the buyer the right, but not the obligation to enter the contract under the terms specified.

Derivatives can be used either for risk management (i.e. to "hedge" by providing offsetting compensation in case of an undesired event, a kind of "insurance") or for speculation (i.e. making a financial "bet"). This distinction is important because the former is a prudent aspect of operations and financial management for many firms across many industries; the latter offers managers and investors a risky opportunity to increase profit, which may not be properly disclosed to stakeholders.

Along with many other financial products and services, derivatives reform is an element of the Dodd–Frank Wall Street Reform and Consumer Protection Act of 2010. The Act delegated many rule-making details of regulatory oversight to the Commodity Futures Trading Commission and those details are not finalized nor fully implemented as of late 2012.

Size of market

To give an idea of the size of the derivative market, The Economist magazine has reported that as of June 2011, the over-the-counter (OTC) derivatives market amounted to approximately $700 trillion, and the size of the market traded on exchanges totaled an additional $83 trillion.[8] However, these are "notional" values, and some economists say that this value greatly exaggerates the market value and the true credit risk faced by the parties involved. For example, in 2010, while the aggregate of OTC derivatives exceeded $600 trillion, the value of the market was estimated much lower, at $21 trillion. The credit risk equivalent of the derivative contracts was estimated at $3.3 trillion.

Still, even these scaled down figures represent huge amounts of money. For perspective, the budget for total expenditure of the United States Government during 2012 was $3.5 trillion, and the total current value of the US stock market is an estimated $23 trillion. The world annual Gross Domestic Product is about $65 trillion.

And for one type of derivative at least, Credit Default Swaps (CDS), for which the inherent risk is considered high, the higher, nominal value, remains relevant. It was this type of derivative that investment magnate Warren Buffet referred to in his famous 2002 speech in which he warned against "weapons of financial mass destruction." CDS notional value in early 2012 amounted to $25.5 trillion, down from $55 trillion in 2008.

8.2 The limits: In mathematics, a **limit** is the value that a function or sequence "approaches" as the input or index approaches some value. Limits are essential to calculus (and mathematical analysis in general) and are used to define continuity, derivatives, and integrals.

The concept of a limit of a sequence is further generalized to the concept of a limit of a topological net, and is closely related to limit and direct limit in category theory. In formulas, a limit is usually denoted "lim" as in $\lim_{n \to c}(a_n) = L$, and the fact of approaching a limit is represented by the right arrow (→) as in $a_n \to L$.

Suppose f is a real-valued function and c is a real number. The expression means that f(x) can be made to be as close to L as desired by making x sufficiently close to c. In that case, the above equation can be read as "the limit of f of x, as x approaches c, is L".

Augustin-Louis Cauchy in 1821,[2] followed by Karl Weierstrass, formalized the definition of the limit of a function as the above definition, which became known as the (ε, δ)-definition of limit in the 19th century. The definition uses ε (the lowercase Greek letter epsilon) to represent any small positive number, so that "f(x) becomes arbitrarily close to L" means that f(x) eventually lies in the interval $(L - \varepsilon, L + \varepsilon)$, which can also be written using the absolute value sign as $|f(x) - L| < \varepsilon$.[2] The phrase "as x approaches c" then indicates that we refer to values of x whose distance from c is less than some positive number δ (the lower case Greek letter delta)—that is, values of x within either $(c - \delta, c)$ or $(c, c + \delta)$, which can be expressed with $0 < |x - c| < \delta$. The first inequality means that the distance between x and c is greater than 0 and that $x \neq c$, while the second indicates that x is within distance δ of c.

Note that the above definition of a limit is true even if $f(c) \neq L$. Indeed, the function f need not even be defined at c.